PROJECTED WRITTEN NOTES FROM THE M408D LECTURE
ON THURSDAY, FEBRUARY 29, 2024,
ON SECTION 10.3: GRAPHING IN POLAR COORDINATES,
AND ON SECTION 10.4: AREAS IN POLAR COORDINATES

CLASS #14

LOG IN TO CANVAS

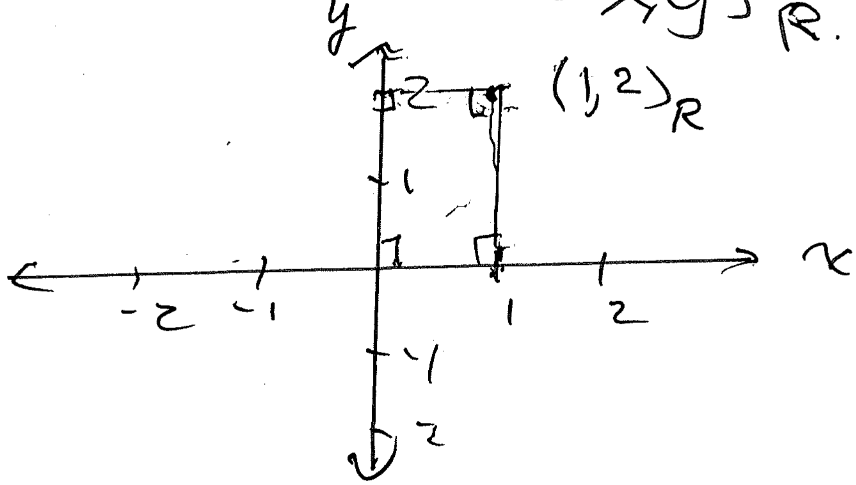
and Read the CANVAS
DAY PAGE FOR TODAY

FOR Tuesday
Read Sections 9.1 and 9.2
and the handouts linked to
on the CANVAS DAY PAGE.

The Polar Coordinate System

Rectangular (Cartesian) Coordinates

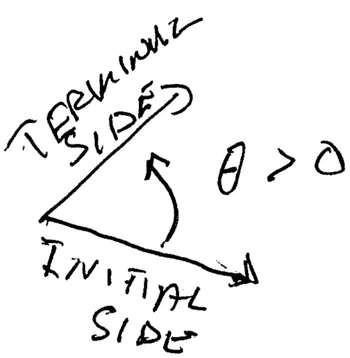
$(x,y)_R$

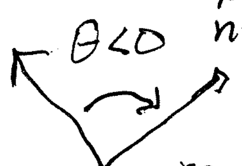


Polar Coordinates $(r,\theta)_P$

A number

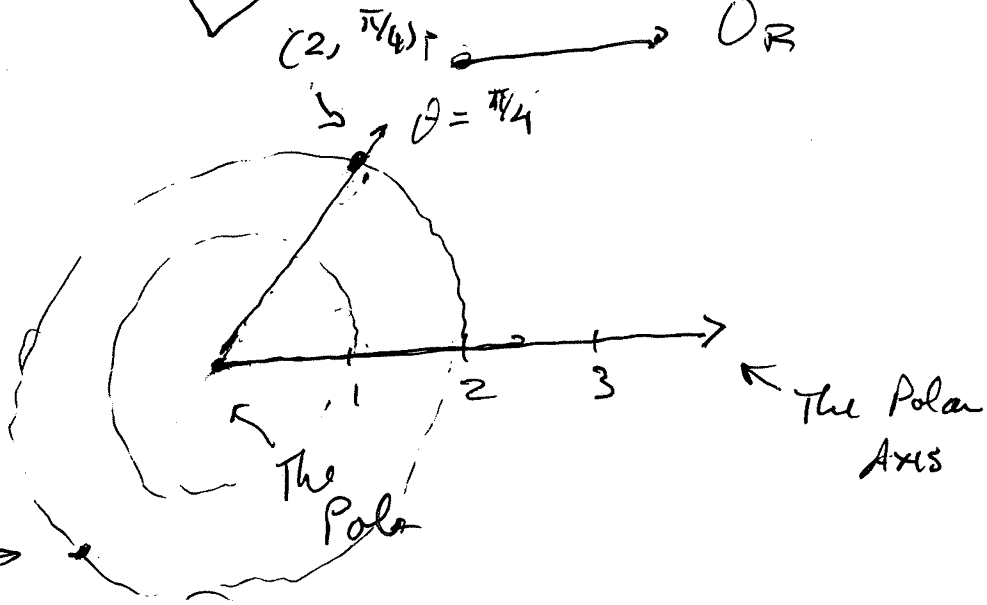
An angle of Rotation





$(r,\theta)_P = (2, \pi/4)$

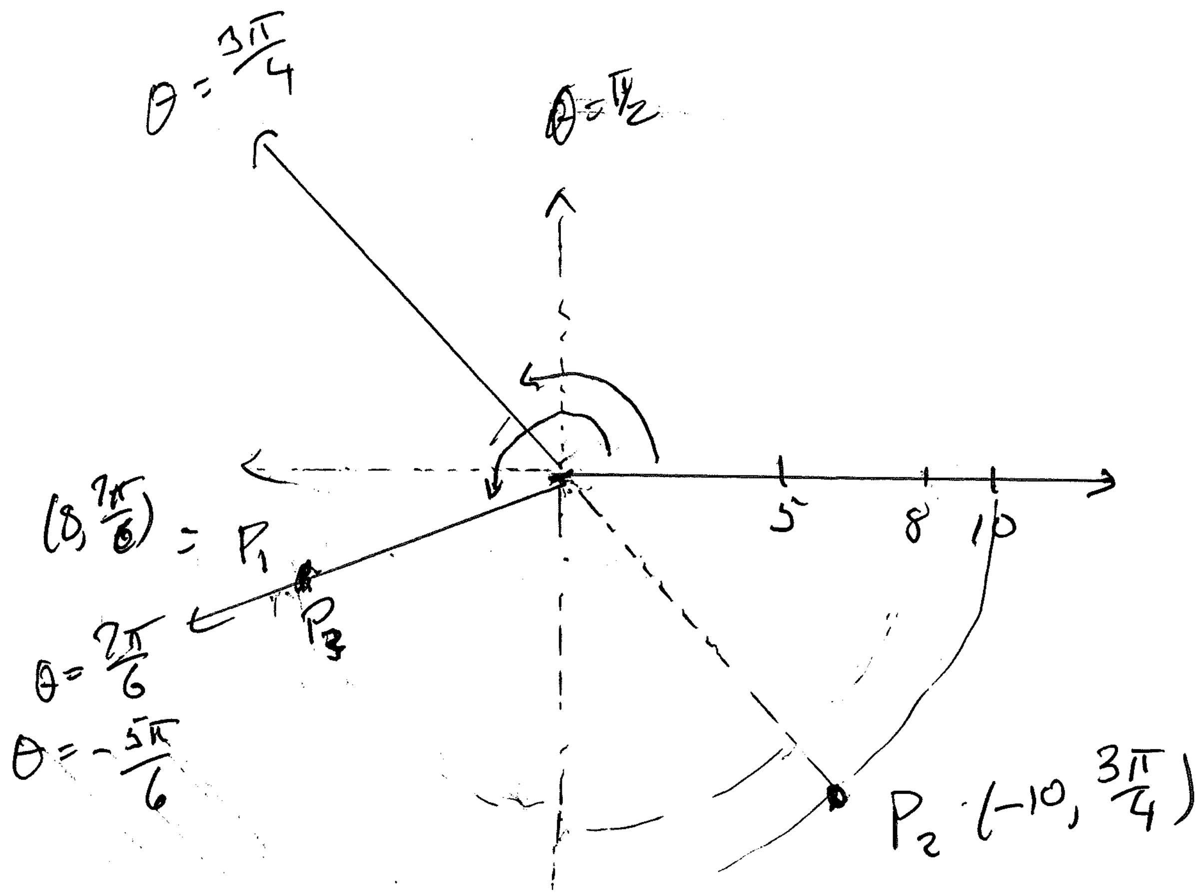
↑ where to look



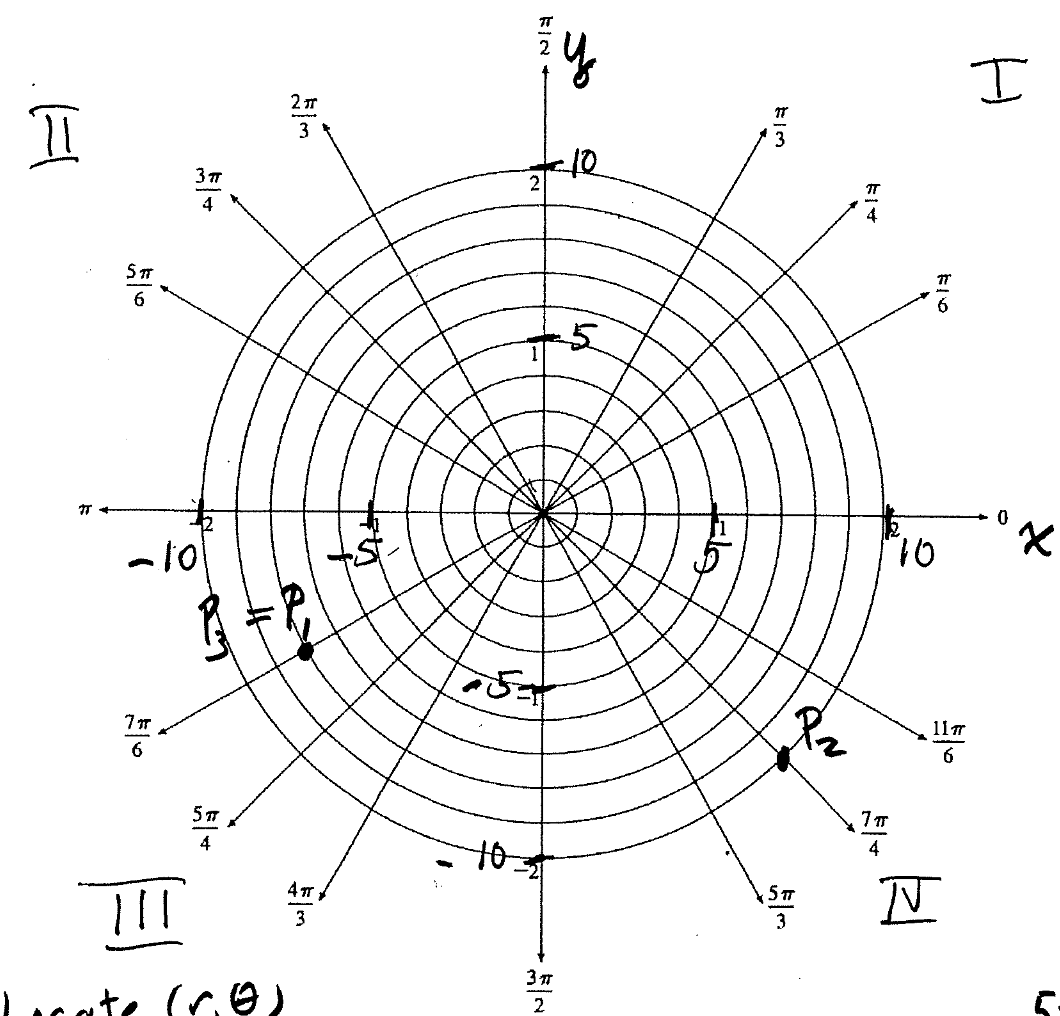
$(r,\theta)_P = (-2, \frac{\pi}{4})_P$

Locate $(r,\theta)_p$

$P_1\ (8, \frac{7\pi}{6})$ $\quad P_2\ (-10, \frac{3\pi}{4})$ $\quad P_3\ (8, -\frac{5\pi}{6})$



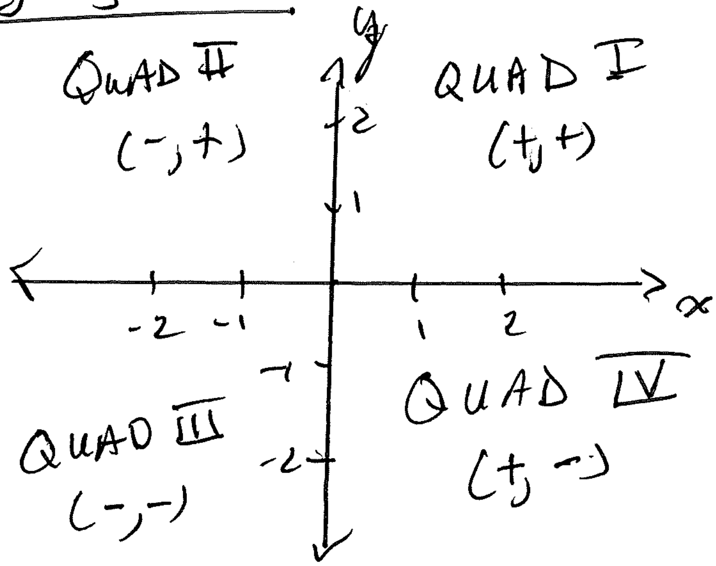
POLAR GRAPH PAPER



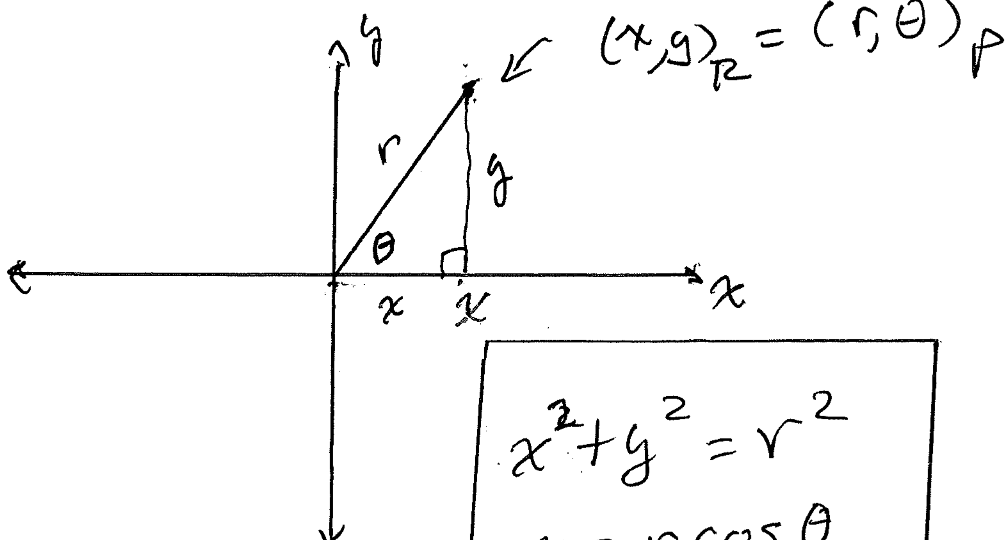
Locate (r, θ)

$P_1 (8, \frac{7\pi}{6})$ $\quad P_2 (-10, \frac{3\pi}{4})$ $\quad P_3 (8, -\frac{5\pi}{6})$

Mixing Systems



Converting Between Systems



$$x^2+y^2=r^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\frac{y}{x} = \tan\theta$$

Polar → Rectangular Conversion

$(r,\theta)_P = (-2, -\pi/4) = (x,y)_R = \underline{(-\sqrt{2}, \sqrt{2})_R}$

$x = -2\cos(-\pi/4) = -2\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$

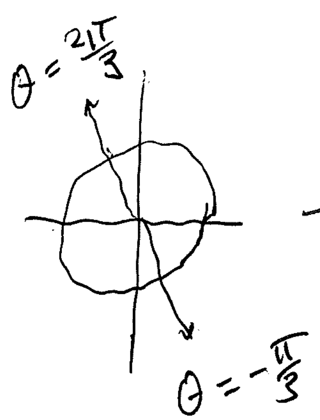
$y = -2\sin(-\pi/4) = -2\left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2}$

Rectangular TO POLAR Conversion

$(x,y)_R = (2, -2\sqrt{3})_R = (r,\theta)_P = \begin{cases} (-4, \frac{2\pi}{3})_P \\ (4, -\frac{\pi}{3})_P \end{cases}$

Get the QUADRANT FIRST!

Here, QUAD is $(+,-)$, QUAD IV



$\frac{y}{x} = \tan\theta = \frac{-2\sqrt{3}}{2} = -\sqrt{3} = \tan\theta$

Point is in QUAD IV

$\theta = \frac{2\pi}{3}$ OR $\theta = -\pi/3$

$x^2+y^2 = r^2 = 4+12 = 16 = r^2$

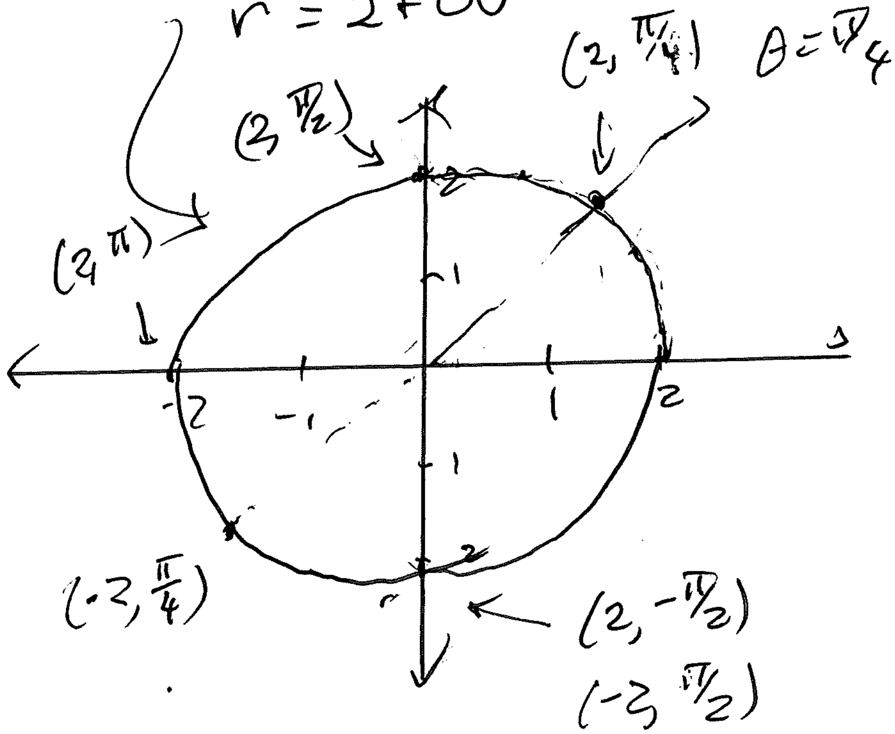
$r = 4$ or $r = -4$

GRAPHS of POLAR EQUATIONS

$r = f(\theta)$
Like, $r = 1 + \cos\theta$

The GRAPH of $r = f(\theta)$ $[F(r,\theta) = 0]$ is the set of points (r,θ) such that each point on the curve has at least one polar coordinate pair that satisfies the equation.

Ex:
FIND THE POLARGRAPH
of $r = 2$
$r = 2 + 0\theta$



$r = -2$
TOO

GRAPHING POLAR (r, θ) EQUATIONS

method ① MAKE A TABLE OF VALUES of (r, θ) for various values of θ. Plot these points and connect them.

SEE EXAMPLE 6 on p 687 of STEWART

method ②

Draw a Rectangular $(\theta, r)_R$ (x, y) of the equation. Interpret each point on the rectangular graph as a line in a table giving the point $(r, \theta)_P$.

SEE EXAMPLE 7 on p. 688 and Figures 12 + 13 on p. 689

EXAMPLE 6

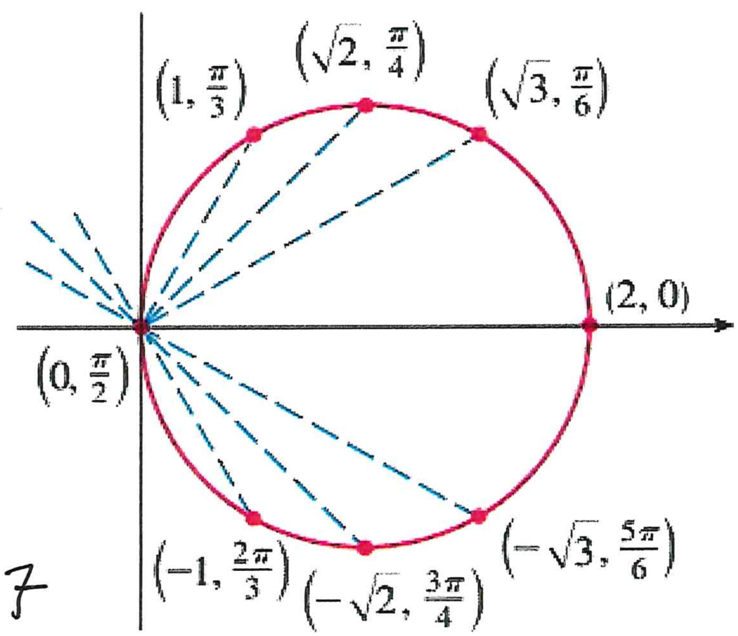
(a) Sketch the curve with polar equation $r = 2\cos\theta$.

(b) Find a Cartesian equation for this curve.

SOLUTION

(a) In Figure 8 we find the values of r for some convenient values of θ and plot the corresponding points (r, θ). Then we join these points to sketch the curve, which appears to be a circle. We have used only values of θ between 0 and π, because if we let θ increase beyond π, we obtain the same points again.

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2



p. 687

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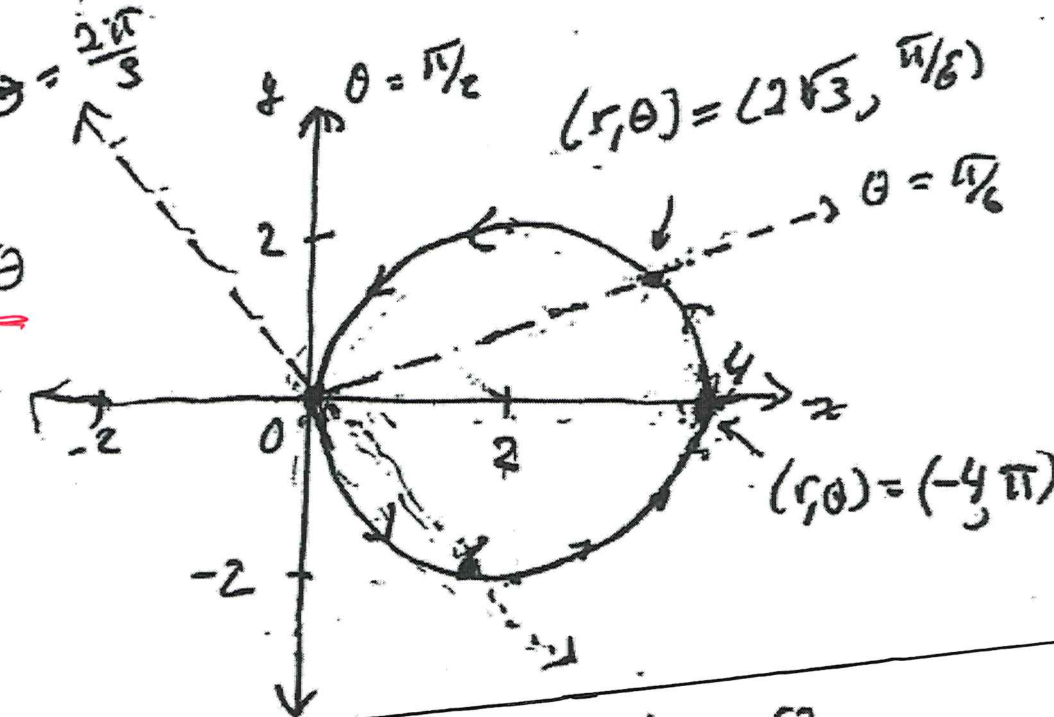
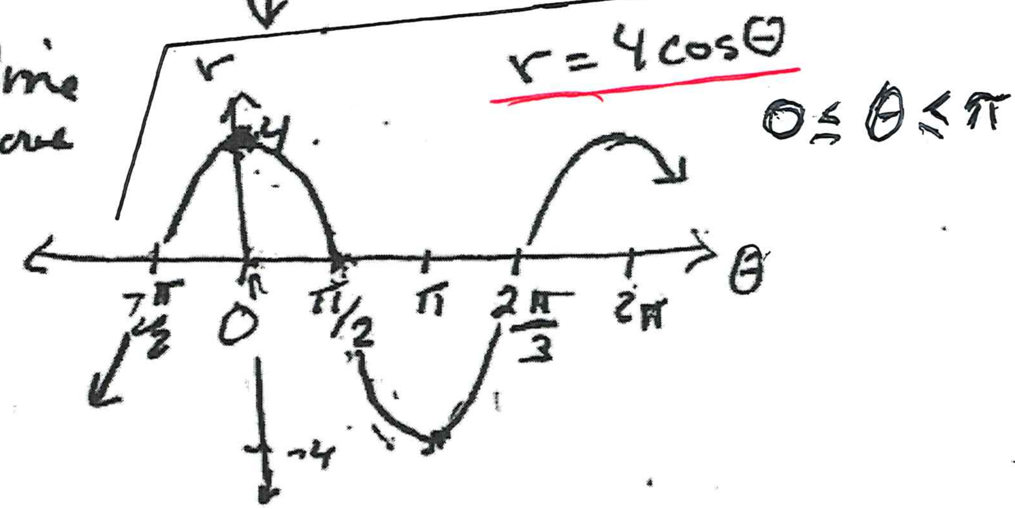
Solution 2:

$r = 4\cos\theta$

Draw a Rectangular (θ, r) graph of this Equation – Each point on the Rectangular graph corresponds to a line in a table like the one above





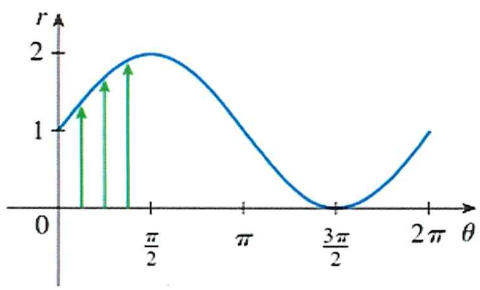
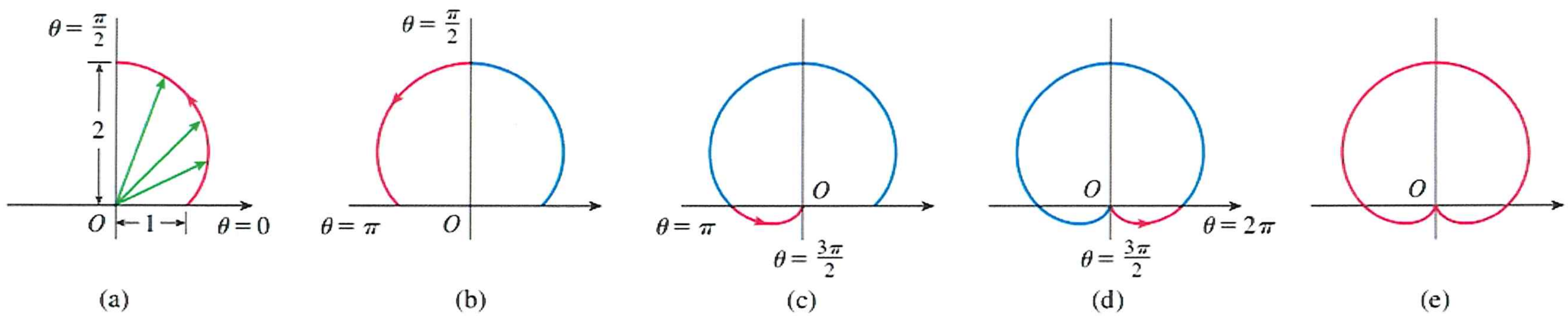


FIGURE 10
$r = 1 + \sin\theta$ in Cartesian coordinates, $0 \leqslant \theta \leqslant 2\pi$

p. 688

EXAMPLE 7 Sketch the curve $r = 1 + \sin\theta$.

SOLUTION Instead of plotting points as in Example 6, we first sketch the graph of $r = 1 + \sin\theta$ in *Cartesian* coordinates in Figure 10 by shifting the sine curve up one unit. This enables us to read at a glance the values of r that correspond to increasing values of θ. For instance, we see that as θ increases from 0 to $\pi/2$, r (the distance from O) increases from 1 to 2 (see the corresponding green arrows in Figures 10 and 11), so we sketch the corresponding part of the polar curve in Figure 11(a). As θ increases from $\pi/2$ to π, Figure 10 shows that r decreases from 2 to 1, so we sketch the next part of the curve as in Figure 11(b). As θ increases from π to $3\pi/2$, r decreases from 1 to 0 as shown in part (c). Finally, as θ increases from $3\pi/2$ to 2π, r increases from 0 to 1 as shown in part (d). If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace this path. Putting together the parts of the curve from Figure 11(a)–(d), we sketch the complete curve in part (e). It is called a **cardioid** because it's shaped like a heart.



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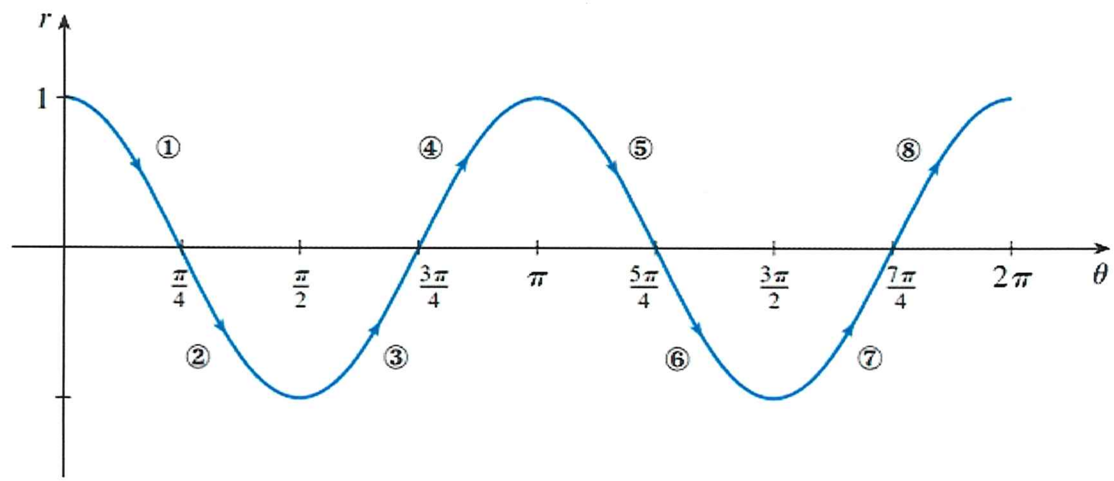

FIGURE 12
$r = \cos 2\theta$ in Cartesian coordinates

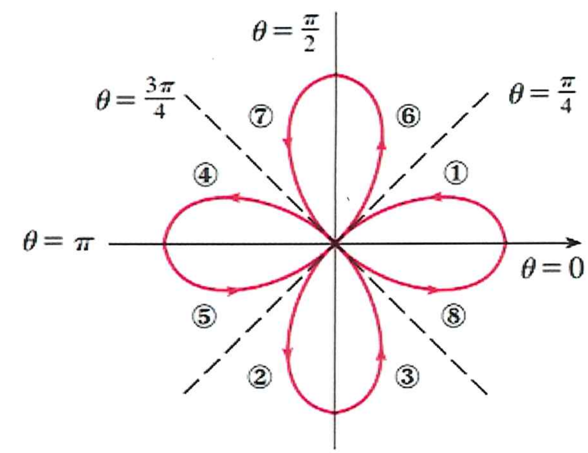

FIGURE 13
Four-leaved rose $r = \cos 2\theta$

p. 689

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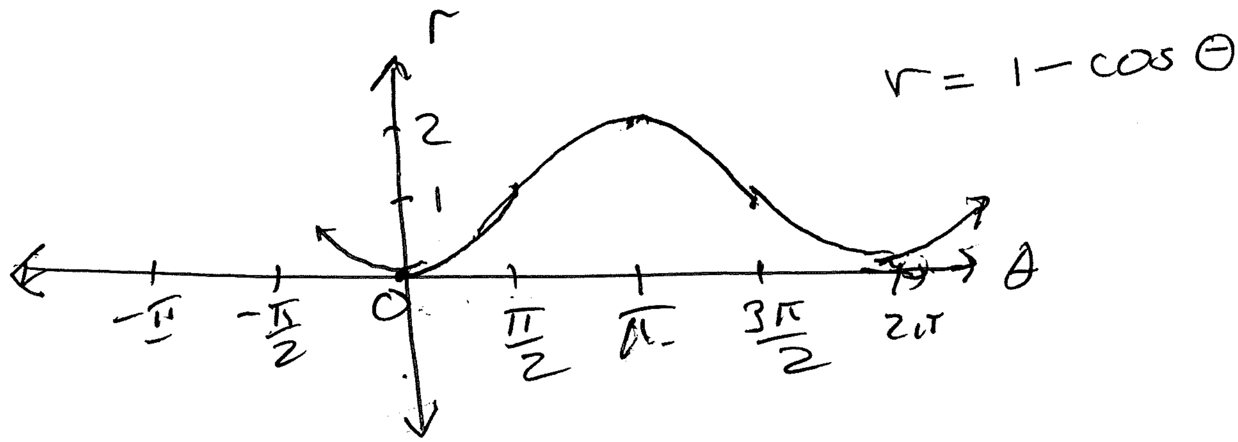
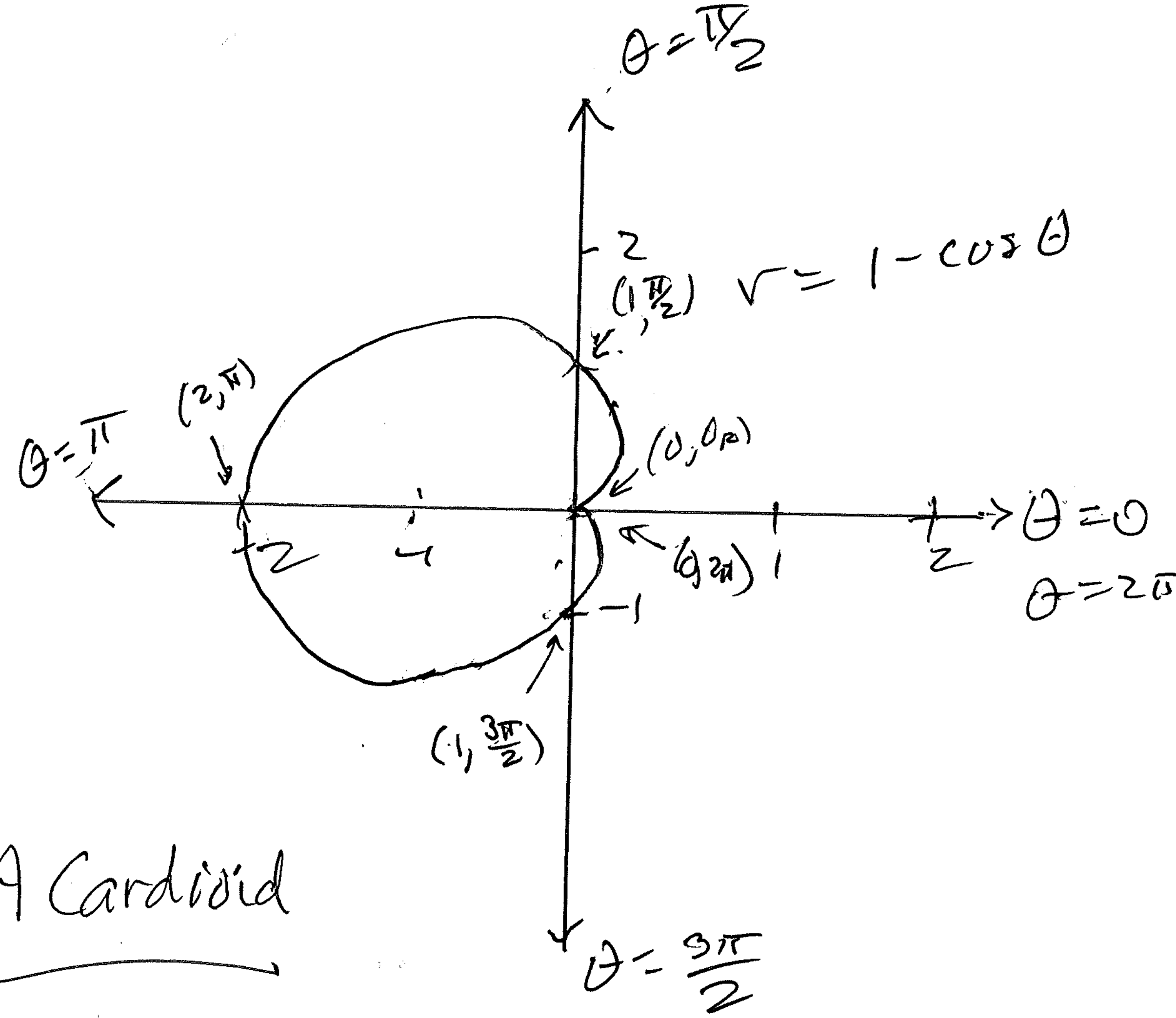
GRAPH The Polar Equation $r = 1 - \cos\theta$

$0 \le \theta \le 2\pi$



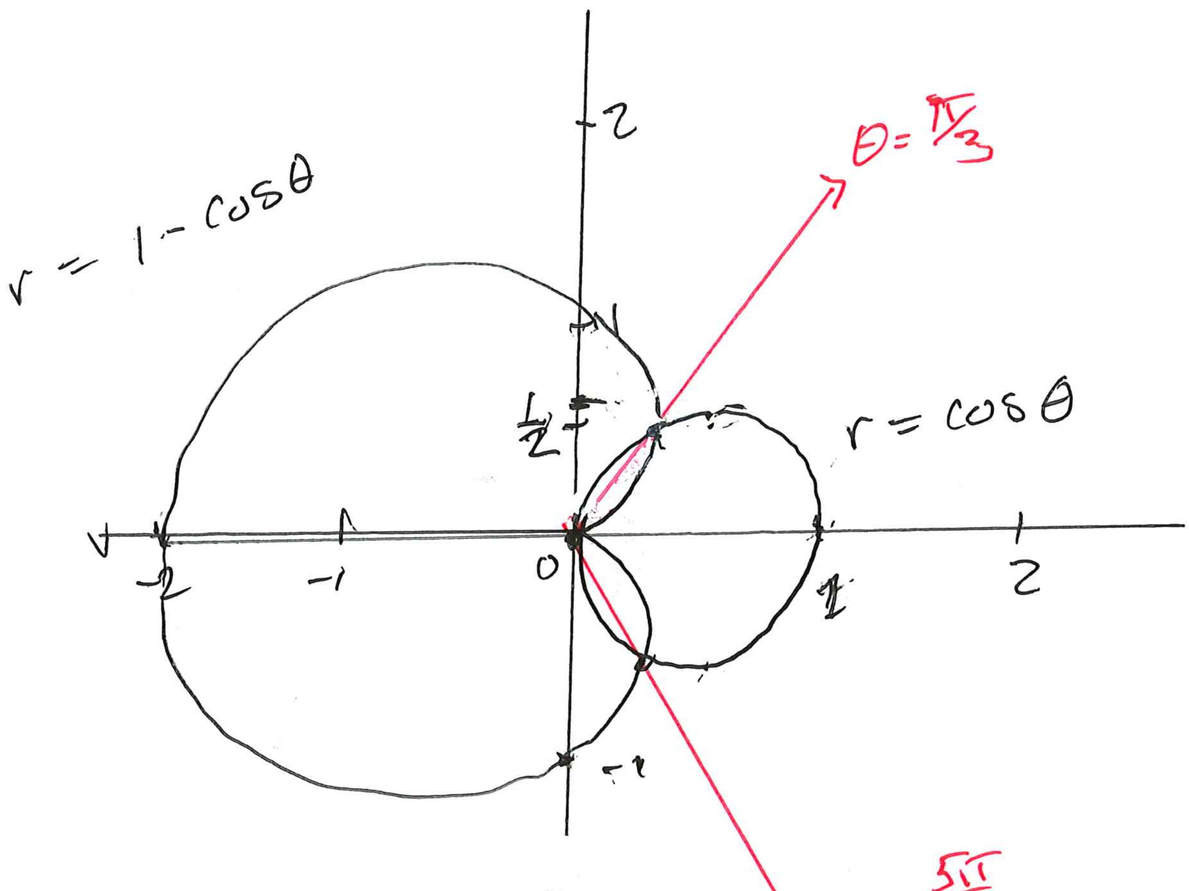


A Cardioid

Problem: Find the points of intersection of the two graphs

① $r = 1 - \cos\theta$

② $r = \cos\theta$



Step 1: Draw the Graphs $\qquad \theta = \frac{5\pi}{3}$

Step 2: Set the expressions for r equal to each other. $r = 1 - \cos\theta = \cos\theta$

$\therefore\ r = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$

$r = \cos\frac{\pi}{3} = \frac{1}{2} \Rightarrow \left(\frac{1}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{1}{2}, \frac{5\pi}{3}\right)$

$\left(\frac{1}{2}, -\frac{\pi}{3}\right)$

STEP 3: CHECK if The POLE is on Both Graphs

Set $r = 0$

on ①: $r = 1 - \cos\theta = 0 \Rightarrow 1 = \cos\theta \Rightarrow \theta = 0$. $\theta = 2\pi$

on ②: $r = \cos\theta = 0 \Rightarrow \theta = \pi/2$

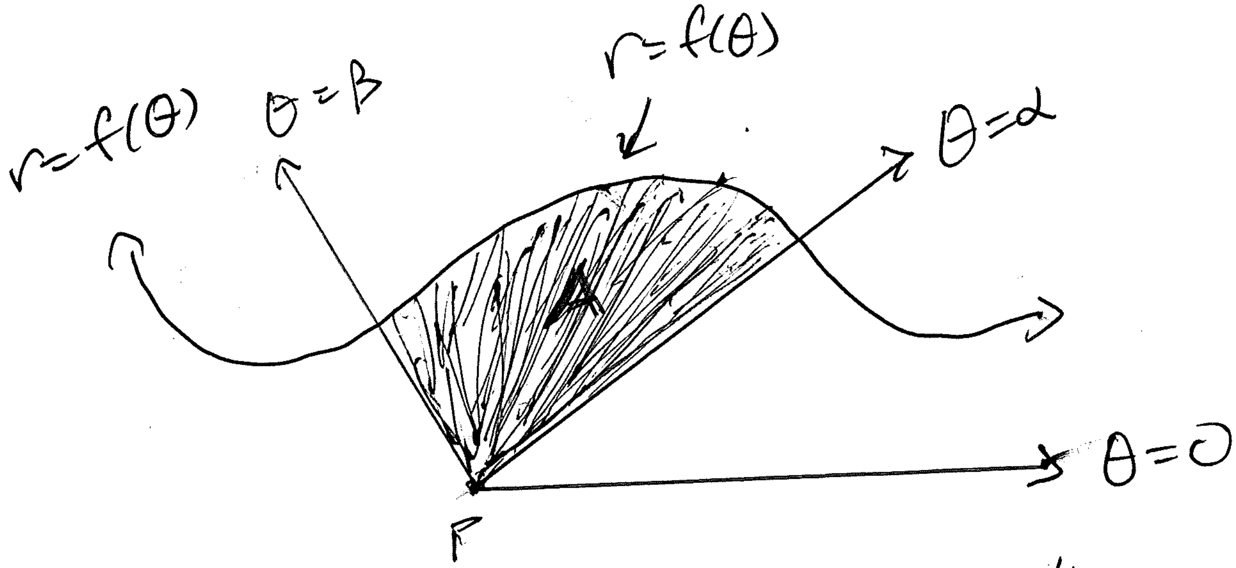
$\theta = \frac{3\pi}{2}$

Yes, The Pole is on both Graphs.

The Points of intersection are

$(\frac{1}{2}, \frac{\pi}{3})$, $(\frac{1}{2}, \frac{5\pi}{3})$, $(0, 0)$

The Area A "inside" a Polar Curve

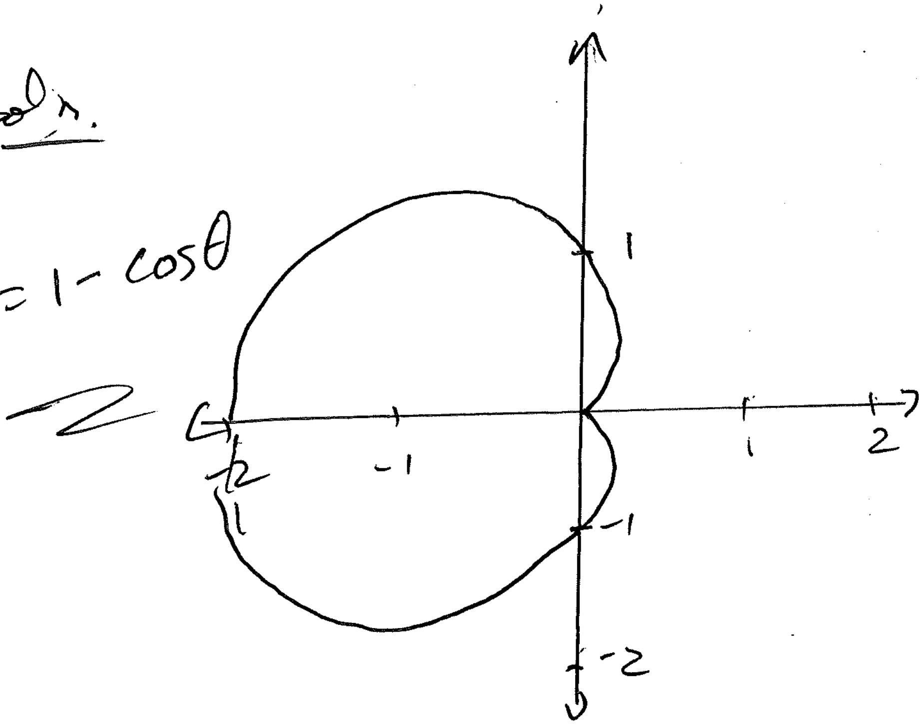


The Area A of the region "inside"
for $\alpha \le \theta \le \beta$
is
$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left[f(\theta)\right]^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2\, d\theta$$

Problem: Find the Area A inside the Cardiod $r = 1 - \cos\Theta$.

Soln.

$r = 1 - \cos\theta$



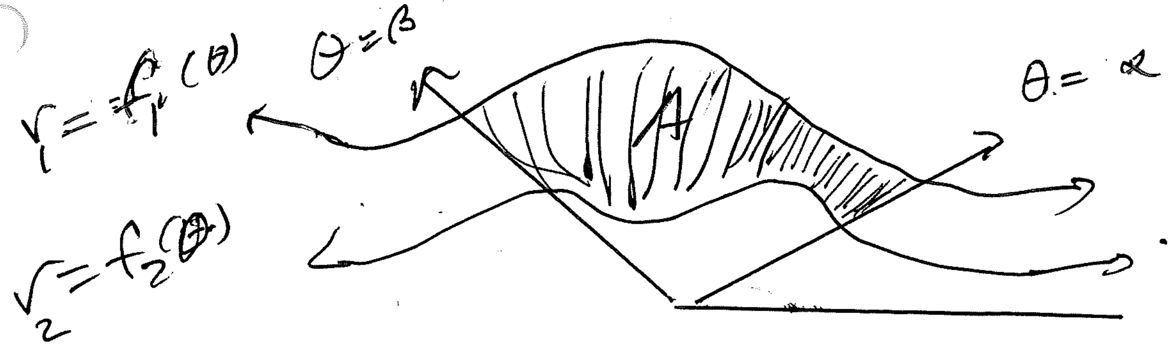
$$\text{Area } A = \int_0^{2\pi} \frac{1}{2}\left[1-\cos\theta\right]^2 d\theta = 2\int_0^{\pi} \frac{1}{2}(1-\cos\theta)^2 d\theta$$

using Symmetry

$$= \dots = A = \frac{3\pi}{2} \text{ Square units}$$

$$\approx 4.712 \text{ units}^2$$

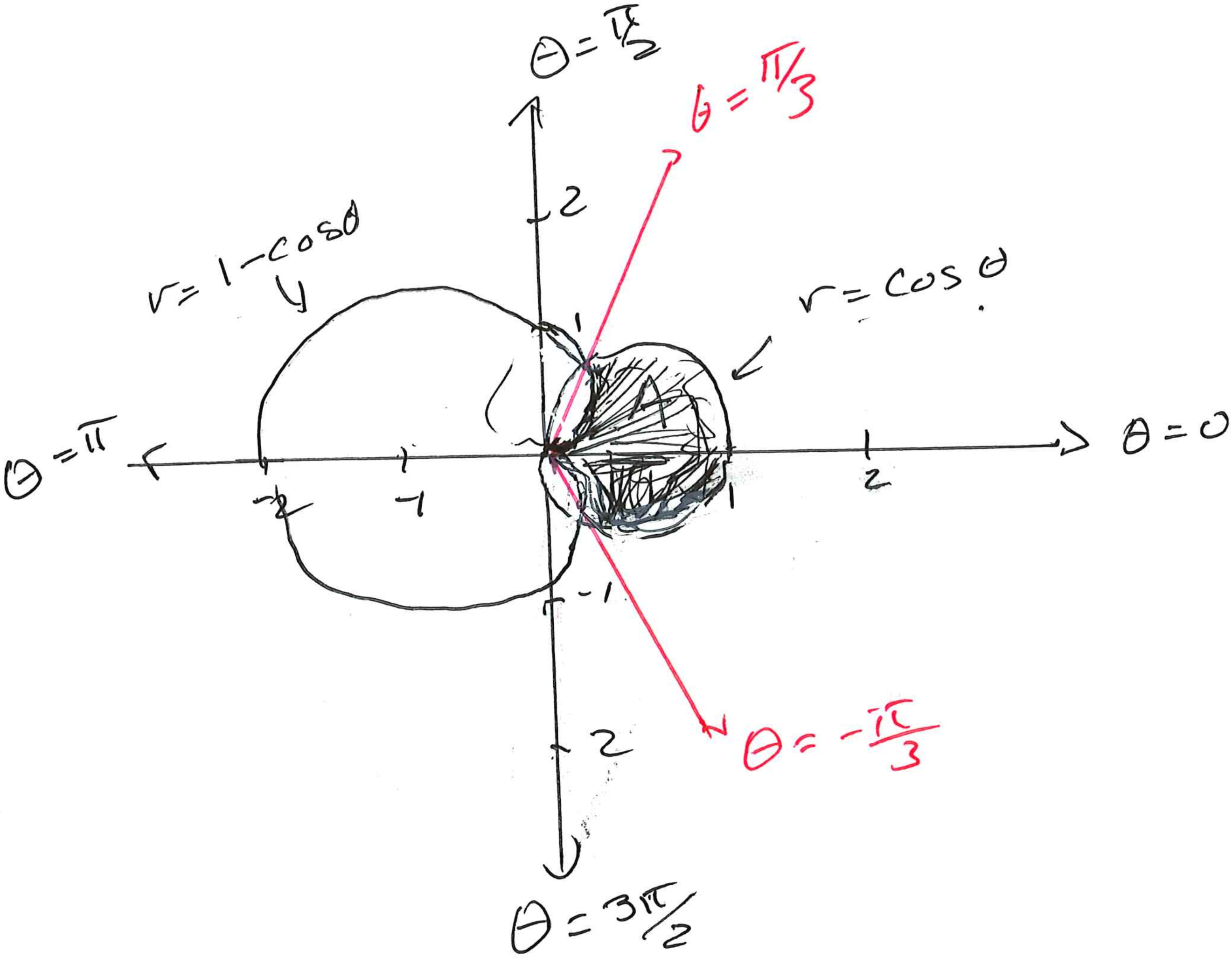
For the Area Between two Polar Curves:



$$\text{Area } A = \int_{\alpha}^{\beta} \frac{1}{2}\left[r_1^2\right] d\theta - \int_{\alpha}^{\beta} \frac{1}{2}\left[r_2^2\right] d\theta$$

$$\text{Area } A = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_1^2 - r_2^2\right) d\theta$$

Problem : Find the Area A of the region that is inside the circle $r = \cos\theta$ but is outside the Cardioid $r = 1 - \cos\theta$



$$\text{Area } A = \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left((\cos\theta)^2 - (1-\cos\theta)^2 \right) d\theta$$

$$= \ldots \sqrt{3} - \pi/3 \text{ Sq. units}$$

$$\approx \underline{0.6849 \text{ units}^2}$$

$$\text{AREA } A = \int_{-\frac{\pi}{3}}^{\pi/3} \frac{1}{2}\left((\cos\theta)^2 - (1-\cos\theta)^2 \right) d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left[\cos^2\theta - (1 - 2\cos\theta + \cos^2\theta) \right] d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left[\cancel{\cos^2\theta} - 1 + 2\cos\theta - \cancel{\cos^2\theta} \right] d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \frac{1}{2}(2\cos\theta - 1)\, d\theta = \int_{-\pi/3}^{\pi/3} \left(\cos\theta - \frac{1}{2}\right) d\theta$$

$$= \left(\sin\theta - \frac{1}{2}\right)\Big]_{-\pi/3}^{\pi/3} = \left(\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) - \left(-\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right)\right.$$

$$= 2\,\frac{\sqrt{3}}{2} - 2\,\frac{\pi}{6}$$

$$= \sqrt{3} - \frac{\pi}{3} \text{ sq. units} \approx 0.6849 \text{ units}^2$$